Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. The exam has 5 questions with equal weight. Some questions have multiple parts. Point totals are given in the margin.

Student Id: Answer Key

1.	20
2.	20
3.	20
4.	20
5.	20
Total	100

No notes or books may be used on the exam. If you have any questions, please raise your hand and I will try to answer them.

1. Consider drawing two cards from a standard deck of 52 cards.

$\boxed{5}$ (a) What is the probability that both cards are aces?

Answer:

$$\binom{4}{2} \Big/ \binom{52}{2} \approx 0.0045$$

$\boxed{5}$ (b) What is the probability that at least one card is an ace?

Answer:

$$\left(\binom{4}{1}\binom{48}{1} + \binom{4}{2}\right) \Big/ \binom{52}{2} \approx 0.1493$$

$\boxed{5}$ (c) What is the probability that both cards are aces given that at least one card is an ace?

Answer:

$$\frac{\binom{4}{2}/\binom{52}{2}}{\left(\binom{4}{1}\binom{48}{1} + \binom{4}{2}\right)/\binom{52}{2}} = \frac{6}{198} = \frac{1}{33} \approx 0.0303$$

$\boxed{5}$ (d) What is the probability that both cards are aces given that one card is the ace of spades?

Answer:

$$\frac{\binom{1}{1}\binom{3}{1}/\binom{52}{2}}{\binom{1}{1}\binom{51}{1}/\binom{52}{2}} = \frac{3}{51} = \frac{1}{17} \approx 0.0588$$

2. A system consists of 4 components, each of which operates correctly with probability p. If a subsystem consists of components connected in *series*, then all must work correctly for the subsystem to work correctly. If a subsystem consists of components connected in *parallel*, then at least one must work correctly for the subsystem to work correctly. Assume that each component works correctly or fails independently.

$\boxed{5}$ (a) If the components are all connected in series, what is the probability that the entire system works?

Answer:

$$p^4$$

$\boxed{5}$ (b) If the components are all connected in parallel, what is the probability that the entire system works?

Answer:

$$1-(1-p)^4$$

$\boxed{5}$ (c) If the components are all connected in parallel, what is the probability that the system fails given that one of the components has failed?

Answer:

$$(1-p)^3$$

$\boxed{5}$ (d) If a subsystem of two components connected in parallel is constructed, and then two such subsystems are connected in series to form the system, what is the probability that the entire system works?

Answer:

$$(1-(1-p)^2)^2$$

3. A group consists of 6 men and 4 women. Assume a person is equally likely to be born on any day of the week.

5 (a) What is the probability that exactly 2 people are born on a Sunday?

Answer:

$$\binom{10}{2}\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^8 \approx 0.2676$$

5 (b) What is the probability that at most three men are born on a weekday?

Answer:

$$\sum_{i=0}^{3}\binom{6}{i}\left(\frac{5}{7}\right)^i\left(\frac{2}{7}\right)^{6-i} \approx 0.2297$$

5 (c) What is the probability that exactly 1 man and exactly 1 woman are born on a Sunday?

Answer:

$$\binom{6}{1}\left(\frac{1}{7}\right)\left(\frac{6}{7}\right)^5 \cdot \binom{4}{1}\left(\frac{1}{7}\right)\left(\frac{6}{7}\right)^3 \approx 0.1427$$

5 (d) What is the probability that all of the women are born on different days of the week?

Answer:

$$\left(\frac{7}{7}\right)\left(\frac{6}{7}\right)\left(\frac{5}{7}\right)\left(\frac{4}{7}\right) \approx 0.3499$$

4. Answer the following questions:

$\boxed{5}$ (a) Two friends play a shooting game, and the probability that friend i hits the target is p_i. At each round, if only one hits the target, then that friend is the winner and the game ends; otherwise, they play another round. What is the expected number of rounds played?

Answer:

$$\frac{1}{p_1(1-p_2)+(1-p_1)p_2}$$

$\boxed{5}$ (b) In grading 25 exams, a teaching assistant has made errors on 4 different exams. If the professor randomly selects 5 different exams, what is the probability that the professor finds at least half of the errors?

Answer:

$$\sum_{i=2}^{4} \frac{\binom{4}{i}\binom{21}{5-i}}{\binom{25}{5}} \approx 0.1664$$

$\boxed{5}$ (c) If it is estimated that 2 compact discs manufactured in a given month by a company are defective, what is the probability that next month at most two defective discs will be made?

Answer:

$$e^{-2}+2e^{-2}+2e^{-2}=5e^{-2}\approx 0.6767$$

$\boxed{5}$ (d) Assuming that n is sufficiently large, what is the approximate probability using a Poisson distribution that out of n people, somebody else has the same birthday as you?

Answer:

$$1-e^{-n/365}$$

[10] 5. (a) Prove the following identity for a discrete random variable X, where $\mu = E[X]$:

$$E[(X-\mu)^2] = E[X^2] - E[X]^2$$

Answer:

$$\begin{aligned}
E[(X-\mu)^2] &= \sum_x (x-\mu)^2 p(x) \\
&= \sum_x (x^2 - 2x\mu + \mu^2) p(x) \\
&= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) \\
&= E[X^2] - 2\mu E[X] + \mu^2 \\
&= E[X^2] - 2E[X]^2 + E[X]^2 \\
&= E[X^2] - E[X]^2
\end{aligned}$$

[10] (b) Let X be a binomial random variable with parameters n and p. Prove that $E[X^2] = np(E[Y]+1)$, where Y is a binomial random variable with parameters $n-1$ and p.

Answer:

$$\begin{aligned}
E[X^2] &= \sum_{i=0}^{n} i^2 p(i) \\
&= \sum_{i=0}^{n} i^2 \binom{n}{i} p^i q^{n-i} \\
&= \sum_{i=1}^{n} i^2 \binom{n}{i} p^i q^{n-i} \\
&= \sum_{i=1}^{n} in \binom{n-1}{i-1} p^i q^{n-i} \\
&= n \sum_{i=1}^{n} i \binom{n-1}{i-1} p^i q^{n-i} \\
&= n \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p p^j q^{(n-1)-j} \\
&= np \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j q^{(n-1)-j} \\
&= np \left[\sum_{j=0}^{n-1} j \binom{n-1}{j} p^j q^{(n-1)-j} + \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} \right] \\
&= np(E[Y]+1)
\end{aligned}$$