Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. The exam has 5 questions with equal weight. Some questions have multiple parts. Point totals are given in the margin.

Student Id: Answer Key

1.	20
2.	20
3.	20
4.	20
5.	20
Total	100

No notes or books may be used on the exam. If you have any questions, please raise your hand and I will try to answer them.

1. A committee is to be chosen from a set of 7 women (one of whom is Mrs. Smith) and 4 men (one of whom is Mr. Smith). How many ways are there to form the committee under each of the following conditions?

5 (a) The committee has 5 people, 3 women and 2 men.

Answer:

$$\binom{7}{3}\binom{4}{2} = 210$$

5 (b) The committee can be any size (except empty), but it must have equal numbers of women and men.

Answer:

$$\binom{7}{1}\binom{4}{1} + \binom{7}{2}\binom{4}{2} + \binom{7}{3}\binom{4}{3} + \binom{7}{4}\binom{4}{4} = 329$$

5 (c) The committee has 4 people and 1 of them must be Mr. Smith.

Answer:

$$\binom{10}{3} = 120$$

5 (d) The committee has 4 people, 2 of each sex, and Mr. and Mrs. Smith cannot both be on the committee.

Answer:

$$\binom{6}{2}\binom{3}{1} + \binom{6}{1}\binom{3}{2} + \binom{6}{2}\binom{3}{2} = 108$$

2. Consider the following combinatorial identity ($0 \leq i \leq j \leq n$):

$$\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$$

10 (a) Prove the identity algebraically using the factorial definition of a combination.

Answer:

$$\binom{n}{j}\binom{j}{i} = \frac{n!}{j!(n-j)!}\frac{j!}{i!(j-i)!} = \frac{n!}{(n-j)!(j-i)!i!}$$

$$\binom{n}{i}\binom{n-i}{j-i} = \frac{n!}{i!(n-i)!}\frac{(n-i)!}{(j-i)!(n-i-(j-i))!} = \frac{n!}{i!(j-i)!(n-i-j+i)!} = \frac{n!}{(n-j)!(j-i)!i!}$$

10 (b) Prove the identity combinatorially. *Hint:* From a set of n people a committee of size j is to be chosen, and from this committee a subcommittee of size $i \leq j$ is also to be chosen. One can choose the committee first or the subcommittee first.

Answer: There are $\binom{n}{j}$ ways to choose the committee of size j from n people and $\binom{j}{i}$ ways to choose the subcommittee of size i from the committee. By the basic principle of counting, there are $\binom{n}{j}\binom{j}{i}$ ways to form both the committee and the subcommittee.

Alternatively, there are $\binom{n}{i}$ ways to form the subcommittee of size i from n people and $\binom{n-i}{j-i}$ ways to choose the remaining $j-i$ members of the committee of size j from the remaining $n-i$ people. By the basic principle of counting, there are $\binom{n}{i}\binom{n-i}{j-i}$ ways to form both the committee and the subcommittee.

3. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected.

5 (a) What is the probability that the committee will have 1 student from each class?

Answer:

$$\binom{3}{1}\binom{4}{1}\binom{4}{1}\binom{3}{1}/\binom{14}{4} = \frac{144}{1001} \approx .1439$$

5 (b) What is the probability that the committee will have 2 sophomores and 2 juniors?

Answer:

$$\binom{4}{2}\binom{4}{2}/\binom{14}{4} = \frac{36}{1001} \approx .0360$$

5 (c) What is the probability that the committee will have only sophomores and juniors?

Answer:

$$\binom{8}{4}/\binom{14}{4} = \frac{70}{1001} \approx .0699$$

5 (d) What is the probability that the committee will have all sophomores or all juniors?

Answer:

$$\left[\binom{4}{4} + \binom{4}{4}\right]/\binom{14}{4} = \frac{2}{1001} \approx .0020$$

4. A sports league consists of 16 teams divided into 4 divisions of 4 teams each. The team with the best record (assume no ties occur) in each division advances to the playoffs. From the remaining teams, the 2 teams with the next best records regardless of division (again, assume no ties occur) are called wildcard teams and also advance.

10 (a) What is the probability that both wildcard teams come from the same division?

Answer:

$$\binom{4}{1}\binom{3}{2}/\binom{12}{2} = \frac{12}{66} \approx .1818$$

10 (b) What is the probability that the wildcard teams come from different divisions?

Answer:

$$\binom{4}{2}\binom{3}{1}\binom{3}{1}/\binom{12}{2} = \frac{54}{66} = 1 - \frac{12}{66} \approx .8182$$

5 5. (a) Suppose we play poker dice by rolling 5 fair dice one at a time. What is the probability of rolling a full house (any three dice having one number and two remaining dice having another number) given that the first 2 dice have different numbers?

Answer:

$$\frac{3!}{6^3} \approx .0278$$

5 (b) What is the probability given that the first 2 dice have the same number?

Answer:

$$\frac{5+5\binom{3}{2}}{6^3} = \frac{20}{6^3} \approx .0926$$

10 (c) A test has been developed to detect a certain disease in individuals. It is known that 10% of all individuals have the disease. The proposed test was given to diseased individuals and a correct result was obtained in 95% of the cases. When the test was given to individuals without the disease, 5% were reported to mistakenly have the disease. What is the probability that an individual has the disease given that the test indicates its presence?

Answer: Let D be the event that an individual has the disease. Let R be the event that the result is positive. Then, $P(D) = .10$, $P(R|D) = .95$, and $P(R|D^c)$ = .05.

$$P(D|R) = \frac{P(DR)}{P(R)} = \frac{P(RD)}{P(R)} = \frac{P(R|D)P(D)}{P(R)} = \frac{P(R|D)P(D)}{P(R|D)P(D)+P(R|D^c)P(D^c)} = \frac{.95\cdot.10}{.95\cdot.10+.05\cdot.90} = \frac{.0950}{.0950+.0450} \approx .6786$$