1. Maps

Recall that a map \( f: M \rightarrow N, \ m \rightarrow f(m) \) is determined by three components, the set \( M \) of arguments, the range set \( N \), and an association rule that associates with each and every element \( m \in M \) a single image \( f(m) \).

Assume that \( M_1 \subset M \) is a subset of \( M \). The restriction \( f|_{M_1}: M_1 \rightarrow N, \ m \rightarrow f(m) \) has the same association rule as \( f \), but a different range of arguments.

**Definition:** A map \( f: M \rightarrow N, \ m \rightarrow f(m) \) is called injective iff the following condition is true:

\[
\forall m_1, m_2 \in M : f(m_1) = f(m_2) \Rightarrow m_1 = m_2
\]

**Remark:** The injectivity condition is equivalent to the following one

\[
\forall m_1, m_2 \in M : m_1 \neq m_2 \Rightarrow f(m_1) \neq f(m_2).
\]

This follows from a general principle of elementary logic. If \( S, T \) are two statements, then the statement \( S \Rightarrow T \) is equivalent to the statement \( -T \Rightarrow -S \).

**Definition:** A map \( f: M \rightarrow N, \ m \rightarrow f(m) \) is called surjective iff

\[
\forall n \in N \ \exists m \in M : f(m) = n.
\]

**Remark:** A map is surjective if every element of the range \( N \) of \( f \) is indeed the image of (one or more elements) of \( M \).

**Definition:** A map \( f: M \rightarrow N, \ m \rightarrow f(m) \) is called bijective iff it is both injective and surjective.

**Definition:** Let \( f: M \rightarrow N, \ m \rightarrow f(m) \) and \( g: N \rightarrow P, \ n \rightarrow g(n) \) be two maps such that the set of arguments of the latter is the range of the former. Then the composition of \( f \) and \( g \) is

\[
g \circ f: M \rightarrow P, \ m \rightarrow g(f(m)).
\]

We call \( \text{Id}_M: M \rightarrow M, \ m \rightarrow \text{Id}_M(m) := m \) the identity map.

**Proposition:** If \( f: M \rightarrow N, \ m \rightarrow f(m) \) is bijective, then there exists a map \( f^{-1}: N \rightarrow M \) such that \( f \circ f^{-1} = \text{Id}_N \) and \( f^{-1} \circ f = \text{Id}_M \).
2. Group Homomorphisms

Definition: Let \((G_1, \cdot)\) and \((G_2, \cdot)\) be two groups, both with a product written as the multiplication “\(\cdot\)”. A group homomorphism \(h: G_1 \to G_2\) is a map that fulfills the following condition:

\[\forall a, b \in G_1 : h(a \cdot b) = h(a) \cdot h(b).\]

We often say: “Let \(h: G_1 \to G_2\) be a group homomorphism” to mean “Let \((G_1, \cdot)\) and \((G_2, \cdot)\) be two groups and let \(h: G_1 \to G_2\) be a group homomorphism.

Proposition: Let \(h: G_1 \to G_2\) be a group homomorphism. Then

1. \(h\) maps the unit element (a.k.a. the identity element or the neutral element) of \(G_1\) into the unit element of \(G_2\).
2. For all \(a \in G_1\) we have \(f(a^{-1}) = (f(a))^{-1}\).

Proposition: Let \(G\) be a commutative group. (That is, the commutative law holds.) Let \(n\) be an integer. Then \(p_n: G \to G, g \mapsto p_n(g) := g^n\) defines a group homomorphism. Similarly, \(i: G \to G, g \mapsto i(g) := g^{-1}\) is a group homomorphism.

Proposition: Let \(h_1: G_1 \to G_2\) and \(h_2: G_2 \to G_3\) be group homomorphisms. Then the composition \(h_2 \circ h_1: G_1 \to G_3\) is also a group homomorphism.

Proposition: Let \(h: G_1 \to G_2\) be a group homomorphism. Define \(\text{kern}(h) = \{x \in G_1 \mid h(x) = e_{G_2}\}\) to be the set of all elements in \(G_1\) that are mapped onto the identity element of the second group. Then \(\text{kern}(h)\) is a subgroup.

Proposition: Let \(h: G_1 \to G_2\) be a group homomorphism. Define \(\text{im}(h) = \{x \in G_2 \mid \exists a \in G_1 : h(a) = x\}\) to be the set of all elements in \(G_2\) that are an image of one or more elements in \(G_1\). Then \(\text{im}(h)\) is a subgroup.

Proposition: Let \(h: G_1 \to G_2\) be a group homomorphism. Then \(h\) is injective iff \(\text{kern}(h) = \{e_{G_1}\}\).