**Definition:** Let $F$ be a field. A set $V$ and two mappings $V \times V \rightarrow V$, $(x, y) \mapsto x + y$ (the addition) and $F \times V \rightarrow V$, $(\alpha, a) \mapsto \alpha a$ (the scalar multiplication) forms a **vector space** over $F$ if the following axioms are valid:

(A1) \quad \forall x, y, z \in V : x + (y + z) = (x + y) + z

(A2) \quad \forall x, y \in V : x + y = y + x

(A3) \quad \exists 0 \in V \forall x \in V : 0 + x = x. \quad \text{(It follows that this element is unique and called the zero)}

(A4) \quad \forall x \in V \exists y \in V : x + y = 0. \quad \text{(It follows that } x \text{ determines } y. \text{ Therefore, we call } y \text{ the negative of } x \text{ and write it as } y = -x.\)

(S1) \quad \forall \alpha, \beta \in F \forall x \in V : \alpha(\beta x) = (\alpha \beta)x.

(S2) \quad \forall x \in V : 1x = x.

(S3) \quad \forall \alpha \in F \forall x, y \in V : \alpha(x + y) = \alpha x + \alpha y.

(S4) \quad \forall \alpha, \beta \in F \forall x \in V : (\alpha + \beta)x = \alpha x + \beta x.

Prove the following rules for all scalars $\alpha, \beta \in F$ and all vectors $x, y \in V$:

1. $0x = 0$. \quad \text{(There are two zeroes, the one in the field and the other one in the vectorspace.)}
2. $\alpha 0 = 0$.
3. $(-\alpha)x = - (\alpha x)$.
4. $(-\alpha)x = \alpha(-x)$.
5. $\alpha x = 0$ implies that $\alpha = 0$ or that $x = 0$.
6. Define $x - y = x + (-y)$. Then $\alpha (x - y) = \alpha x - \alpha y$.

Assume it is known that the set $\mathbb{R}$ of all real numbers is a field. Prove the following:

1. The set $C = \{(x,y) \mid x, y \in \mathbb{R}\}$ is a field if addition is defined as $(a,b) + (c,d) = (a+b, c+d)$ and multiplication defined by $(a,b)(c,d) = (ac-bd, ad + bc)$.
2. Show that the same set with multiplication defined by $(a,b)(c,d) = (ac+bd, ad + bc)$ is not a field.
3. Show that the set $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ with addition defined by $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$ and scalar multiplication defined by $v(x_1, x_2, \ldots, x_n) = (v \cdot x_1, v \cdot x_2, \ldots, v \cdot x_n)$ is a vector space.