(In the following, assume that all vectors are elements of a fixed vector space $V$ over the field $\Phi$.)

**Definition:** A subspace of a vector space $V$ (over field $\Phi$) is a set $S \subseteq V$ such that $S$ is a vector space in its own right.

**Lemma:** A subset $M \subset V$ is a subspace of vector space $V$ if and only if the following conditions hold:
1. $0 \in M$.  
2. For all $a, b \in M$: $a + b \in M$.  
3. For all $a \in M$ and $x \in \Phi$: $xa \in M$.

**Lemma:** If $S$ and $T$ are two subspaces of $V$, then $S \cap T$ is also a subspace.

**Lemma:** If $S$ and $T$ are two subspaces of $V$, then $S \cup T$ is usually not a subspace.

**Definition:** If $v_1, v_2, \ldots, v_n \in V$, then $< v_1, v_2, \ldots, v_n >= \left\{ \sum_{\nu=1}^{n} x_\nu v_\nu \mid x_1, x_2, \ldots, x_n \in \Phi \right\}$ is the linear hull of $v_1, v_2, \ldots, v_n$.

**Definition:** $v_1, v_2, \ldots, v_n \in V$ are called linearly dependent if there exists $x_1, x_2, \ldots, x_n \in \Phi$ not all zero such that $\sum_{\nu=1}^{n} x_\nu v_\nu = 0$. Conversely, $v_1, v_2, \ldots, v_n \in V$ are called linearly independent if $\sum_{\nu=1}^{n} x_\nu v_\nu = 0$ implies that $x_1 = x_2 = \ldots = x_n = 0$.

**Lemma:** If $v_1, v_2, \ldots, v_n \in V$ are linearly independent and $v_{n+1} \not\in < v_1, v_2, \ldots, v_n >$, then $v_1, v_2, \ldots, v_n, v_{n+1}$ are linearly independent, too.

**Definition:** $v_1, v_2, \ldots, v_n \in V$ are a base of $V$ if and only if
1. $< v_1, v_2, \ldots, v_n > = V$.  
2. $v_1, v_2, \ldots, v_n$ are linearly independent.

**Theorem:** If $v_1, v_2, \ldots, v_n \in V$ are a base of $V$ then every vector $v \in V$ can be written uniquely as $\sum_{\nu=1}^{n} x_\nu v_\nu = v$.

Our next task is to prove that the number of elements in a finite base of a vector space are uniquely determined.
Definition: $M = \{v_1, v_2, \ldots, v_n\}$ is a set of generators of $V$ if $V = \langle v_1, v_2, \ldots, v_n \rangle$. We then also say that $M$ generates $V$.

Lemma: The following are equivalent:
1. $M$ is a base of $V$.
2. $M$ is a minimal set of generators for $V$.
3. $M$ generates $V$ and $M$ is linearly independent.
4. $M$ is a maximal linearly independent subset of $V$.

Theorem: Every vector space has a base.