COEN279 - Design and Analysis of Algorithms

Greedy Algorithm

Dr. Tunghwa Wang

Fall 2016

Announcement

- Grader: Xiaoyu Li
  - Email: natalienew2016@gmail.com
  - Future submissions to her: CC to me
    - Filename should contain your name and type of homework: for example, TunghwaWang-Bonus2
    - You may resubmit, same file names: new file replaces the old one.
- Programming assignment #2:
  - Submission:
    - When you run “perl Submit P2”, you should see the message of successful submission to natalienew2016@gmail.com and twang1@cse.scu.edu.

Announcement

- Exams:
  - Grade scheme is designed to allow you work diligently to learn from class.
    - It is just how much you learn from class.
  - Exam is aimed to confirm it.
  - Midterm Exam:
    - Top score: 114
    - Bottom score: 40
    - Average score: 72.4
  - Final Exam:
    - Similar approach: to understand complex algorithm starting from base cases
    - Skill: to abstract complex problem starting from base cases
    - Goal: to make sure of your understanding the algorithms and methods.

Matroids

- A theory describes cases which can be solved using greedy algorithm:
  - It is not true the other way:
    - Not all cases for greedy algorithm are matroids.
    - Neither one of activity selection and Huffman coding presented is.
  - Definition: A matroid is an ordered pair $M = \{ S, \mathscr{I} \}$ satisfying the conditions:
    - S is a finite set.
    - $\mathscr{I}$ is a non-empty set of subsets of S such that if $B \in \mathscr{I}$ and $A \subseteq B$, then $A \in \mathscr{I}$:
      - We call $\mathscr{I}$ an independent subsets of S with some property of being "independent".
      - $\varnothing \in \mathscr{I}$
      - We say that $\mathscr{I}$ is hereditary.
    - If $A \in \mathscr{I}$, $B \in \mathscr{I}$, and $|A| < |B|$, then there exists some element $x \in B\text{-}A$ such that $A \cup \{x\} \in \mathscr{I}$.
      - We say that M satisfies exchange property.

Matroids

- Example:
  - Matric matroid:
    - For a matrix M, let S = { r: r is row vector of M }, then an independent set $A \subseteq S$ is defined as being linearly independent.
      - Obviously, any $B \subseteq A$ still contains only linearly independent vectors of M.
      - If $A \in \mathscr{I}$, $B \in \mathscr{I}$, and $|A| < |B|$, then there exists some element $x \in B\text{-}A$ such that $A \cup \{x\} \in \mathscr{I}$ because at least one row vector can be linearly independent to all row vectors in A.
        - If dimension of M is HxK, that is dimension of $r \in S$ is K.
        - Let A= { r: r is row vector of M } and B= { r: r is row vector of M }, then $|A| < |B| \leq K$: Otherwise, then they cannot be linearly independent.
        - For each $x \in S$, we can try to transform it into $c_1a_1 + c_2a_2 + \ldots + c_{|A|}a_{|A|}$ where c_k is scalar and $a_k \in A$ for all k = 1, 2, …, |A|. There can only be at most |A| such x's from S_B to get transformed: Otherwise, they will not be linearly independent.
        - Thus, whatever x cannot be transformed will be the one making $A \cup \{x\}$ independent set and thus $A \cup \{x\} \in \mathscr{I}$.

Matroids

- Example:
  - Graph matroid:
    - For a graph G = (V, E), let S = E, then an independent set $A \subseteq S$ is defined as being acyclic which is equivalent to G_A = (V, A) being a forest.
      - That is, G' = (V, A) forms a forest.
      - Obviously, any $B \subseteq A$ is still acyclic.
      - If $A \in \mathscr{I}$, $B \in \mathscr{I}$, and |A| < |B|, then there exists some element $x \in B\text{-}A$ such that $A \cup \{x\} \in \mathscr{I}$ because at least one edge does not form cycle to all edges in A.
        - By definition, let G_A contains i connected trees and G_B contains j connected trees.
        - From mathematical induction, for each connected acyclic tree (V_C, E_C), $|E_C| = |V_C| - 1$.
        - Because $|A| = (|V_1| - 1) + (|V_2| - 1) + \ldots + (|V_i| - 1) = |V_A| - i = |V| - i$ and $|B| = (|V_1'| - 1) + (|V_2'| - 1) + \ldots + (|V_j'| - 1) = |V_B| - j = |V| - j$, $|B|-|A| = (|V|-j)-(|V|-i) = i-j > 0$. That is, G_A has more connected trees than G_B. Thus, G_B must have a tree containing an edges with vertices in two different connected trees of G_A. We can certainly choose such edge x to G_A without creating cycle. Thus, $A \cup \{x\} \in \mathscr{I}$.

Matroids

- More:
  - If M = { S, $\mathscr{I}$ } is a matroid and x $\notin$ A, then we call x an extension of A if A $\cup$ {x} $\in$ $\mathscr{I}$.
  - If A is an independent subset in matroid M = { S, $\mathscr{I}$ }, then A is maximal if it has no extensions.
    - All maximal independent subsets in a matroid has the same size.
  - A matroid M = { S, $\mathscr{I}$ } is weighted if it is associated with a weight function w:
    - For every x $\in$ S, w(x) is defined and w(x) > 0.
    - For any A $\subseteq$ S, w(A) = $\text{sum}_{x \in S} w(x)$.

Matroids

- A greedy algorithm for a weighted matroid:
  - Let M = { S, I } and |S| = n.

```
greedy_matroid(M, w)
1     A = ∅
2     sort M.S into monotonically decreasing order by weight w
3     for each x ∈ M.S, taken in monotonically decreasing order by weight w(x)
4          if A U {x} ∈ M.I,
5               A = A U {x}
6     return A
```

  - Time complexity:Θ(n(f(n)+lgn))
    - Line 2: Θ(nlgn)
    - Line 3: loop of Θ(n) iterations
    - Line 4: f(n) to verify that A ∪ {x} is an independent set
  - We claim that greedy choice is safe:
    - If M is a weighted matroid with weight function w, then greedy_matroid(M,w) returns an optimal subset.

Matroids

- A greedy algorithm for a weighted matroid:
  - Optimal construct:
    - If $x \in S$ is the first element chosen by the greedy algorithm for M, then finding maximum-weight independent subset containing x is reduced into finding maximal-weight independent subset of weighted matroid M' = { S', I' } where
      - S' = { y: $y \in S$ such that $\{x, y\} \in I$ }
      - I' = { B: $B \subseteq S-\{x\}$ such that $B \cup \{x\} \in I$ }
      - Weight function w applies to all $y \in S'$
  - Greedy property: Choose first x such that {x} is independent.
    - If $x \in S$ is such element, then there exists an optimal subset A of S containing x.
      - Assuming B is an optimal subset, keeping selecting elements from $y \in B$, then $A = A \cup \{y\}$ is still independent subset from exchange property: until $|A| = |B|$.
      - At the moment, B must have one element x left not added to A. Then A, = $B+\{x\}-\{y\}$, is also maximal subset because $w(A) = w(B)+w(x)-w(y) \geq W(B)$.

Greedoids

- A generalization of Matroids:
  - Set system: A set system is an ordered pair { S, $\mathscr{I}$ } such that $\mathscr{I} \subseteq 2^S$.
  - Accessible set system: An accessible set system M = { S, $\mathscr{I}$ } is a set system such that
    - $\varnothing \in \mathscr{I}$
    - For every non-empty $X \in \mathscr{I}$, there exists an $x \in X$ such that $X-\{x\} \in \mathscr{I}$.
    - Elements $X \in \mathscr{I}$ are called feasible sets of M.
    - A maximal feasible set is call basis.
  - Greedoid: A greedoid { S, $\mathscr{I}$ } is a set system which satisfies:
    - $\varnothing \in \mathscr{I}$
    - If $A \in \mathscr{I}$, $B \in \mathscr{I}$, and $|A| < |B|$, then there exists some element $x \in B-A$ such that $A \cup \{x\} \in \mathscr{I}$.
- A greedoid is an accessible set system.

Greedoids

- A greedy algorithm for a weighted greedroid:
  - Let M = { S, I } and |S| = n.

```
greedy_greedoid(M,w)
1     T = ∅
2     X = M.S
3     while there exists x ∈ X such that T ∪ {x} ∈ M.I do
4           choose x ∈ X satisfying T ∪ {x} ∈ M.I such that
                  w(x) ≥ w(y) for all y ∈ X with T ∪ {y} ∈ M.I
5           T = T ∪ {x}
6           X = X - {x}
7     return T
```

  - Time complexity:$\Theta(n^2 f(n))$
    - Line 3: loop of $\Theta(n)$ iterations
    - Line 4: $nf(n)$ to verify that $A \cup \{x\}$ is a feasible set
  - We claim that greedy choice is safe:
    - If M = { S, $\mathscr{I}$ } is a weighted greedoid with weight function w, then greedy_greedoid(M,w) returns an optimal subset if and only if M satisfies the condition
      - For $A \in \mathscr{I}$, $B \in \mathscr{I}$ with |A| = |B| + 1, there exists an element a in A-B such that $B \cup \{a\} \in \mathscr{I}$ and $A-\{a\} \in \mathscr{I}$.

Greedoids

- A greedy algorithm for greedoids:
  - Set system: A set system is an ordered pair { S, $\mathscr{I}$ } such that $\mathscr{I} \subseteq 2^S$.
  - Accessible set system: An accessible set system M = { S, $\mathscr{I}$ } is a set system such that
    - $\varnothing \in \mathscr{I}$
    - For every non-empty $X \in \mathscr{I}$, there exists an $x \in X$ such that $X-\{x\} \in \mathscr{I}$.
    - Elements $X \in \mathscr{I}$ are called feasible sets of M.
    - A maximal feasible set is call basis.
  - Greedoid: A greedoid { S, $\mathscr{I}$ } is a set system which satisfies:
    - $\varnothing \in \mathscr{I}$
    - If $A \in \mathscr{I}$, $B \in \mathscr{I}$, and $|A| < |B|$, then there exists some element $x \in B-A$ such that $A \cup \{x\} \in \mathscr{I}$.
- A greedoid is an accessible set system.

Backup Slides