Linear Algebra (Review)

COEN140

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Vector

- A length-N vector in real domain can be denoted as

$$\mathbf{v} \in \mathbb{R}^N$$

- Example

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = [v_1, v_2, \cdots, v_N]^T$$

- Vector addition: add element by element

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}, \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_N + b_N \end{bmatrix}$$

Vector

- Scalar: a real or complex number

- Multiplying a vector by a scalar

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \qquad \alpha\mathbf{a} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_N \end{bmatrix}$$

Vector

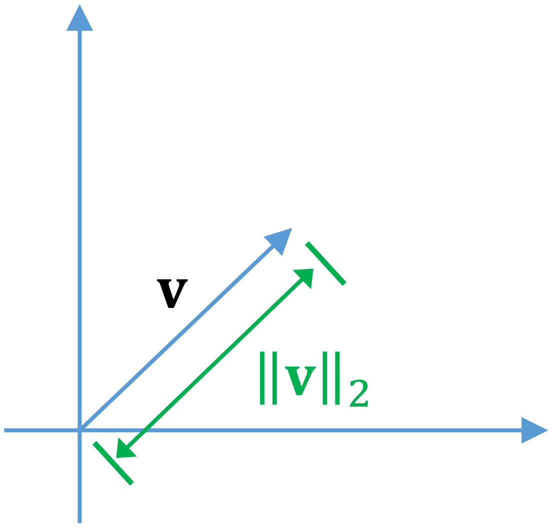
- All-zero vector
  - Column vector: $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [0,0,\ldots,0]^T$
  - Row vector: $\mathbf{0}^T = [0,0,\ldots,0]$

- All-one vector
  - Column vector: $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [1,1,\ldots,1]^T$
  - Row vector: $\mathbf{1}^T = [1,1,\ldots,1]$

Length of a Vector

- For any vector $\mathbf{v} = [v_1, v_2, \cdots, v_N]^T \in \mathbb{R}^N$, its length is defined as

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{i=1}^{N} v_i^2}$$



- $\|\mathbf{v}\|_2$ is also called the L_2-norm of vector $\mathbf{v}$

- If $\|\mathbf{v}\|_2 = 1$, we say the vector $\mathbf{v}$ is normalized, or the vector $\mathbf{v}$ has unit-norm.

Length of a Vector

- What is the L_2-norm of $\frac{\mathbf{v}}{\|\mathbf{v}\|_2}$?

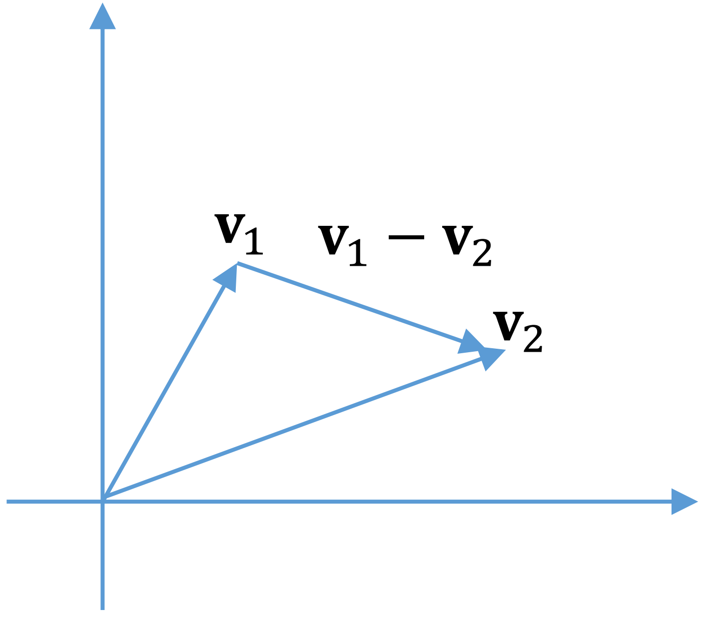
- Answer: 1

Distance Between Vectors

- The Euclidean distance between two vectors in a vector space is defined as

$$d(\mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{v}_1 - \mathbf{v}_2\|_2$$

- Note that $d(\mathbf{v}_1, \mathbf{v}_2) = 0$ iff $\mathbf{v}_1 = \mathbf{v}_2$.

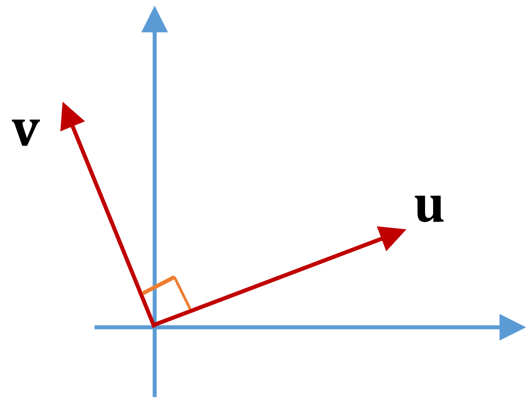


Inner Product Between Vectors

- Define the inner product between two vectors in $\mathbb{R}^N$ by

$$< \mathbf{u}, \mathbf{v} >= \sum_{i=1}^{N} u_i \times v_i = \mathbf{v}^T \mathbf{u} = \mathbf{u}^T \mathbf{v}$$

- Two vectors $\mathbf{u}$ *and* $\mathbf{v}$ are orthogonal when $\mathbf{v}^T \mathbf{u} = 0$



Inner Product Between Vectors

- Calculate the vector L_2-norm of $\mathbf{v} \in \mathbb{R}^N$

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v}^T\mathbf{v}}$$

- The square of the L_2-norm

$$\|\mathbf{v}\|_2^2 = \mathbf{v}^T\mathbf{v} = v_1^2 + v_2^2 + \cdots + v_N^2$$

Matrix

- Matrix: is an array of numbers organized in rows and columns
- Here is a 3×4 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- View a matrix as built from its columns
- $\mathbf{A} = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{a}_4]$

- The k-th column $\mathbf{a}_k = [a_{1k} \;\; a_{2k} \;\; a_{3k}]^T$

Linearly Dependent

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M\}$ are linearly dependent, if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_M$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_M \mathbf{v}_M = \mathbf{0}$$

Linearly Independent

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M\}$ are linearly independent, if the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_M \mathbf{v}_M = \mathbf{0}$$

can only be satisfied by

$$\alpha_1 = \alpha_2 = \cdots = \alpha_M = 0$$

The Rank of a Matrix

- The rank of a matrix is the largest number of linearly independent rows (or columns) in the matrix

- For an $m \times n$ matrix, its rank is $r \leq \min\{m, n\}$

Matrix-Vector Multiplication

- Let $\mathbf{a} = [a_1 \,, a_2 \,, \ldots, a_N \,]^T$
- Let $\mathbf{x} = [x_1 \,, x_2 \,, \ldots, x_N \,]^T$
- $\mathbf{x}^T\mathbf{a} = \mathbf{a}^T\mathbf{x}$: a scalar

- Let $\mathbf{B}$ be an $N \times N$ matrix

- $\mathbf{Bx}$: a column vector

- $\mathbf{x}^T\mathbf{B}$: a row vector

Matrix-Vector Multiplication

- Let $\mathbf{a} = [a_1 \ , a_2 \ , \ldots , a_N \ ]^T$
- Let $\mathbf{x} = [x_1 \ , x_2 \ , \ldots , x_N \ ]^T$
- Let $\mathbf{B}$ be an $N \times N$ matrix

- $\mathbf{x}^T \mathbf{B} \mathbf{x}$ is a scalar

Partial Derivative

- $\mathbf{x}^T\mathbf{a} = \mathbf{a}^T\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_Nx_N$

$$\frac{\partial \mathbf{x}^T\mathbf{a}}{\partial x_i} = a_i$$

- For example

$$\frac{\partial \mathbf{x}^T\mathbf{a}}{\partial x_2} = a_2$$

Vector Derivative

- $\mathbf{x}^T\mathbf{a} = \mathbf{a}^T\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_Nx_N$

$$\frac{d\mathbf{a}^T\mathbf{x}}{d\mathbf{x}} = \frac{d\mathbf{x}^T\mathbf{a}}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{x}^T\mathbf{a}}{\partial x_1} \\ \frac{\partial \mathbf{x}^T\mathbf{a}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{x}^T\mathbf{a}}{\partial x_N} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \mathbf{a}$$

Identity Matrix

- $\mathbf{I}_{D\times D}$: a $D \times D$ identity matrix, a square matrix

- $\mathbf{I}_{D\times D} = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$

- $\mathbf{IA} = \mathbf{A}$
- $\mathbf{AI} = \mathbf{A}$
- $\mathbf{AIB} = \mathbf{AB}$

Transpose

- $\mathbf{x}_{D\times 1}$: a vector

- $\mathbf{x}\mathbf{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_D \end{bmatrix} \times \begin{bmatrix} x_1 & x_2 \dots & x_D \end{bmatrix}$

- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- $(\mathbf{x}\mathbf{x}^T)^T = \mathbf{x}\mathbf{x}^T$: symmetric
- $\mathbf{x}^T\mathbf{x} = \mathbf{x}^T\mathbf{I}\,\mathbf{x}$, where $\mathbf{I}$ is a $D \times D$ identity matrix

Transpose

- $\mathbf{X}_{m \times n}$: a matrix

- $\mathbf{X}\mathbf{X}^T$: $m \times m$

$$(\mathbf{X}\mathbf{X}^T)^T = \mathbf{X}\mathbf{X}^T$$

- $\mathbf{X}^T\mathbf{X}$: $n \times n$

$$(\mathbf{X}^T\mathbf{X})^T = \mathbf{X}^T\mathbf{X}$$

- Both are symmetric

Vector Derivative

- Let $\mathbf{B}$: an $N \times N$ square matrix
- $\mathbf{x} = [x_1 \ , x_2 \ , \ldots, x_N \ ]^T$
- $f(\mathbf{x}) = \mathbf{x}^T\mathbf{B}\mathbf{x} = f\left(x_1 \ , x_2 \ , \ldots, x_N \ \right)$ is a function

$$\frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{d\mathbf{x}} = \begin{bmatrix} \frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{dx_1} \\ \frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{dx_2} \\ \vdots \\ \frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{dx_N} \end{bmatrix} = \mathbf{B}\mathbf{x} + \mathbf{B}^T\mathbf{x}$$

If $\mathbf{B} = \mathbf{B}^T$, then $\frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{d\mathbf{x}} = 2\mathbf{B}\mathbf{x}$

Vector Derivative

- If $\mathbf{B} = \mathbf{B}^T$, then $\frac{d\mathbf{x}^T\mathbf{B}\mathbf{x}}{d\mathbf{x}} = 2\mathbf{B}\mathbf{x}$

- Calculate $\frac{d\mathbf{x}^T\mathbf{x}}{d\mathbf{x}}$?
- $\frac{d\mathbf{x}^T\mathbf{x}}{d\mathbf{x}} = \frac{d\mathbf{x}^T\mathbf{I}\mathbf{x}}{d\mathbf{x}} = 2\mathbf{I}\mathbf{x} = 2\mathbf{x}$

Matrix Inversion

- If $\mathbf{A}_{D\times D}$ is a full-rank square matrix, then
- $\mathbf{A}^{-1}$ exists

- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Matrix Inversion

- If $\mathbf{A}_{D\times D}$ is a full-rank square matrix, then
- $\mathbf{A}^{-1}$ exists

- If $\mathbf{y} = \mathbf{A}\mathbf{x}$, then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

- If $\mathbf{y}^T = \mathbf{x}^T\mathbf{A}$, then

$$\mathbf{x}^T = \mathbf{y}^T\mathbf{A}^{-1}$$

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $\mathbf{I}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix