Probability Review

CSEN266

Santa Clara University

Random Variable

- Basic Idea
  - Don't know how to completely determine what value will occur.
  - Can only specify probabilities of values occurring.

- Examples
  - Number on rolled dice
  - Temperature at specified time of day
  - Friday night attendance at a cinema

Two Types of Random Variable

- Discrete RVs
  - The number of rolled dice
  - Friday night attendance at a cinema

- Continuous RVs
  - Temperature
  - Blood pressure

Notation

- RV: upper case letter
  - e.g. X
- The value of an RV: lower case letter
  - e.g. $X = x$
- Upper case P: the probability, or probability mass function (PMF) of a discrete RV
  - e.g. $P(X = x)$
- Lower case p, or f: the probability density function (PDF) of a continuous RV
  - e.g. $p(x), f(x)$

Discrete Random Variable

- Probability Mass Function (PMF)
- $P(X = x_i) \geq 0, i = 1, 2, \ldots, N$
- $\sum_{i=1}^{N} P(X = x_i) = 1$

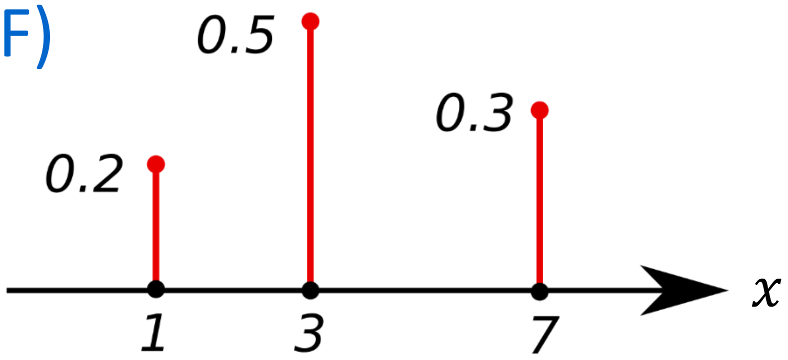


Figure 1

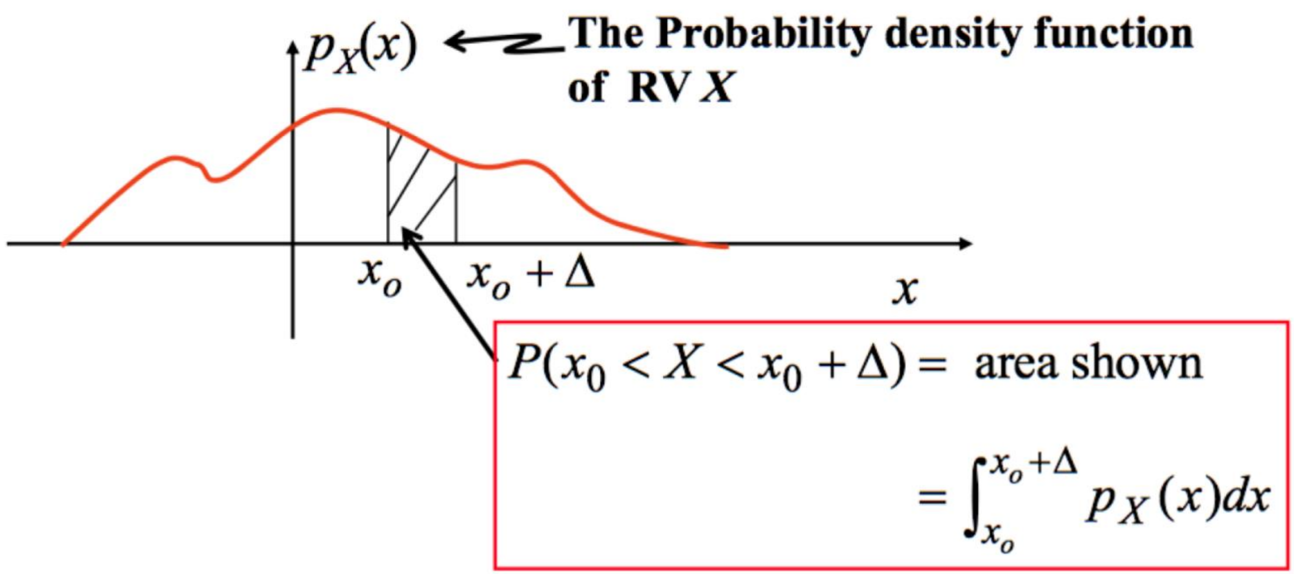
- Cumulative Distribution Function (CDF)
- $F(x) = P(X \leq x)$
- Example in Figure 1
  - $F(1) = P(X \leq 1) = 0.2$
  - $F(3) = P(X \leq 3) = 0.2 + 0.5 = 0.7$
  - $F(4) = P(X \leq 4) =?$
    $=0.2 + 0.5 = 0.7$

Continuous Random Variable

- Given continuous RV X

  Want to know $P(X = x_0)$?

  $P(X = x_0) = 0$
- Because the probability cannot sum up to infinity
- Use probability density function (PDF) to describe the distribution of the continuous RV X



The Mean (or Expectation) of a RV

- For discrete RVs
- $\mu_X = E\{X\} = \sum_{i=1}^{N} x_i P(X = x_i)$

- For continuous RVs
- $\mu_X = E\{X\} = \int_{-\infty}^{+\infty} x \cdot p_X(x) dx$

Example

- Consider two independent coin tosses, each with a ¾ probability of a head, and let X be the number of heads obtained
- Its PMF is

$$P(X=k) = \begin{cases} (1/4)^2 & \text{if } k=0, \\ 2\cdot(1/4)\cdot(3/4) & \text{if } k=1, \\ (3/4)^2 & \text{if } k=2, \end{cases}$$

- Its expectation is

$$E(X) = \sum_k k \times P(X=k) = 0\cdot\left(\frac{1}{4}\right)^2 + 1\cdot\left(2\cdot\frac{1}{4}\cdot\frac{3}{4}\right) + 2\cdot\left(\frac{3}{4}\right)^2 = \frac{24}{16} = \frac{3}{2}.$$

The Variance of a RV

- Variance: characterizes how much the RV deviates from the mean
- Discrete RV

$$\sigma^2 = E\{(X - \mu_X)^2\} = \sum_{i=1}^{N} (x_i - \mu_X)^2 \times P(X = x_i)$$

- Continuous RV

$$\sigma^2 = E\{(X - \mu_X)^2\} = \int_{-\infty}^{+\infty} (x - \mu_X)^2 \cdot p_X(x) dx$$

Joint Probability

- Two RVs
- $P(X = x, Y = y) = P(X = x) \times P(Y = y|X = x)$

$$= P(Y = y) \times P(X = x|Y = y)$$

- If RVs X and Y are independent, then

$$P(X = x, Y = y) = P(X = x) \times P(Y = y)$$

Example

- We toss a fair coin two successive times. A and B are the events:

  A={the 1st toss is a head}

  B = {the 2nd toss is a tail}

- Find $P(A, B)$

Answer:

Events A and B are independent

$$P(A, B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Conditional Probability

- $P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$

- $P(Y = y|X = x) = \frac{P(X=x,Y=y)}{P(X=x)}$

Example

- We toss a fair coin three successive times. A and B are the events:

  A={more heads than tails come up}

  B = {1st toss is a head}

- Find the probability $P(A)$

- Answer:
- Total number of combinations: $2^3 = 8$
- Event A: HHT, HTH, THH, HHH
- $P(A) = \frac{4}{8} = \frac{1}{2}$

Example

- We toss a fair coin three successive times. A and B are the events:

  A={more heads than tails come up}

  B = {1st toss is a head}

- Find the probability $P(B)$

- Answer:
- Total number of combinations: $2^3 = 8$
- Event B: HHH,HHT,HTH,HTT
- $P(B) = \frac{4}{8} = \frac{1}{2}$

Example

- We toss a fair coin three successive times. A and B are the events:

  A={more heads than tails come up}

  B = {1st toss is a head}
- Find the probability $P(A, B)$

- Answer:
- Event A: HHT, HTH, THH, HHH
- Event B: HHH,HHT,HTH,HTT
- $P(A, B) = P(A) \times P(B|A) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$
- Events A and B are not independent

Joint Probability of N RV's

- Chain Rule

- $P(X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N)$

$$= P(X_1 = x_1) \times P(X_2 = x_2 | X_1 = x_1)$$
$$\times P(X_3 = x_3 | X_1 = x_1, X_2 = x_2) \times \cdots$$
$$\times P(X_N = x_N | X_1 = x_1, X_2 = x_2, \cdots, X_{N-1} = x_{N-1})$$

Joint Probability of N RV's

- If RVs $X_1, X_2, \dots, X_N$ are independent, then

$$P(X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N)$$
$$= P(X_1 = x_1) \times P(X_2 = x_2) \times \cdots \times P(X_N = x_N)$$

$$= \prod_{i=1}^{N} P(X_i = x_i)$$

Total Probability

- Assume the observed data point is: $X = x$
- The observation may come from one of N hypotheses (classes): $H_i, i = 1, \ldots, N$

- $P(X = x)$
  $= P(H_1)P(X = x|H_1) + P(H_2)P(X = x|H_2) + \cdots$
  $+ P(H_N)P(X = x|H_N)$

- $P(X = x) = \sum_{i=1}^{N} P(H_i)P(X = x|H_i)$

Example

- You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?

Let A_i be the event of playing with an opponent of type i. We have

$P(A_1) = 0.5, P(A_2) = 0.25, P(A_3) = 0.25$

Let also B be the event of winning. We have

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Thus, by the total probability theorem, the probability of winning is

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3) \\ &= 0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5 \\ &= 0.375.\end{aligned}$$

Bayes' Rule

- Prior Probability
- $P(H_i)$: the probability of hypothesis H_i
- Posteriori probability
- $P(H_i|X = x)$: the probability of hypothesis H_i given the observation that $X = x$

- Bayes Rule

$$P(H_i|X = x) = \frac{P(H_i)P(X = x|H_i)}{P(X = x)}$$

where $P(X = x) = \sum_{i=1}^{N} P(H_i)P(X = x|H_i)$

Example

- Let us return to the chess problem. Here A_i is the event of getting an opponent of type i, and

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, B is the event of winning, and

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Suppose that you win. What is the probability $\mathbf{P}(A_1 \mid B)$ that you had an opponent of type 1?

Using Bayes' rule, we have

$$\begin{aligned}\mathbf{P}(A_1 \mid B) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)} \\ &= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} \\ &= 0.4.\end{aligned}$$